


Linear Regression

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UCDAVIS COMPREHENSIVE CANCER CENTER
UCDAVIS ENVIRONMENTAL HEALTH SCIENCES CENTER

We are video recording this seminar so please hold questions until the end.

Thanks

When to Use Linear Regression

- Continuous outcome variable

- Continuous or categorical predictors

*Need at least one continuous predictor for name “regression” to apply

When NOT to Use Linear Regression

- Binary outcomes
- Count outcomes
- Unordered categorical outcomes
- Ordered categorical outcomes with few (<7) levels

Generalized linear models and other special methods exist for these settings

Some Interchangeable Terms

- Outcome
- Response
- Dependent Variable

- Predictor
- Covariate
- Independent Variable

Simple Linear Regression

Simple Linear Regression

- Model outcome Y by one continuous predictor X:

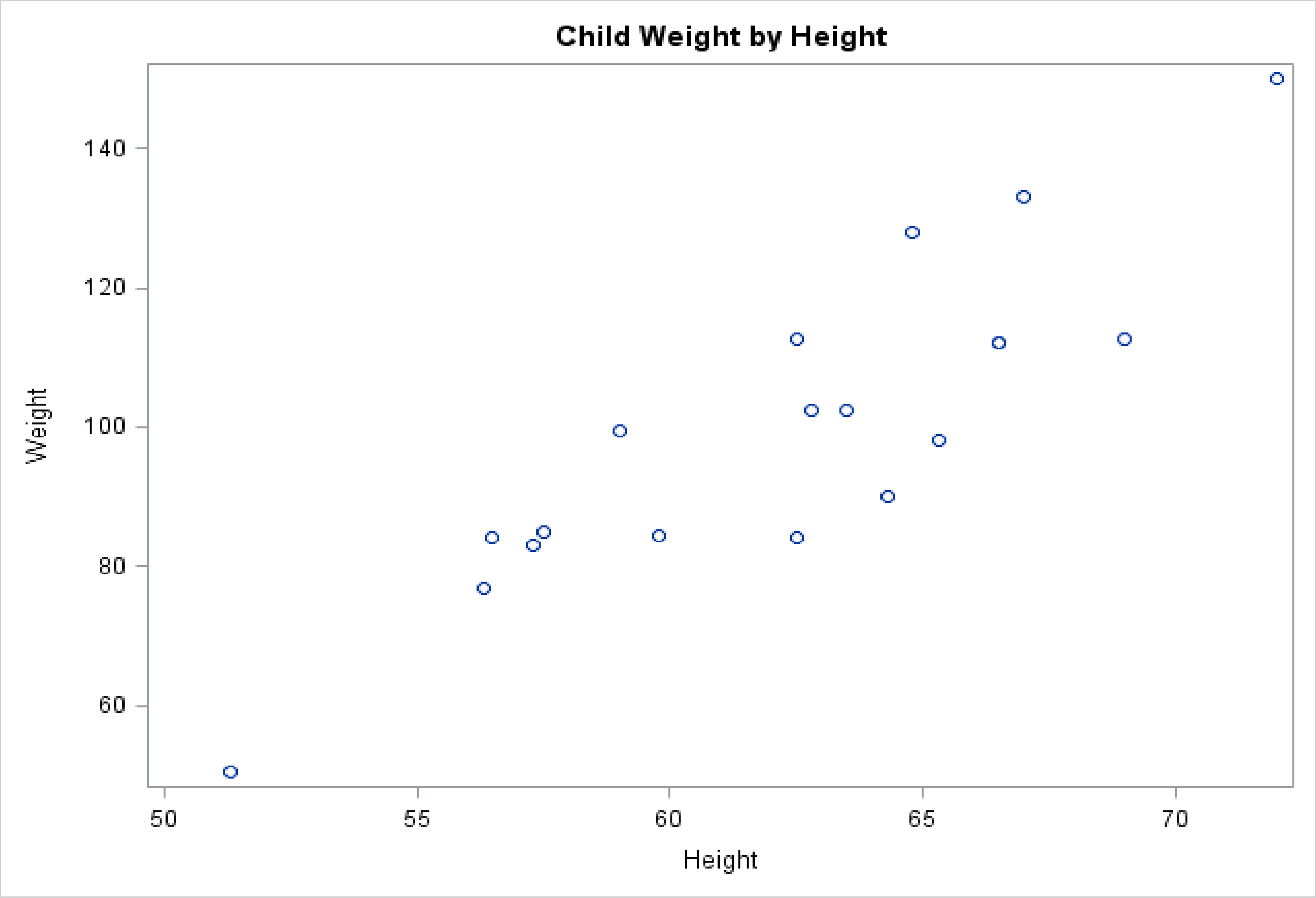
$$Y = \beta_0 + \beta_1 X + \varepsilon$$

- ε is a normally distributed (Gaussian) error term

Model Assumptions

- Normally distributed residuals ε
- Error variance is the same for all observations
- Y is linearly related to X
- Y observations are not correlated with each other
- X is treated as fixed, no distributional assumptions
- Covariates do not need to be normally distributed!

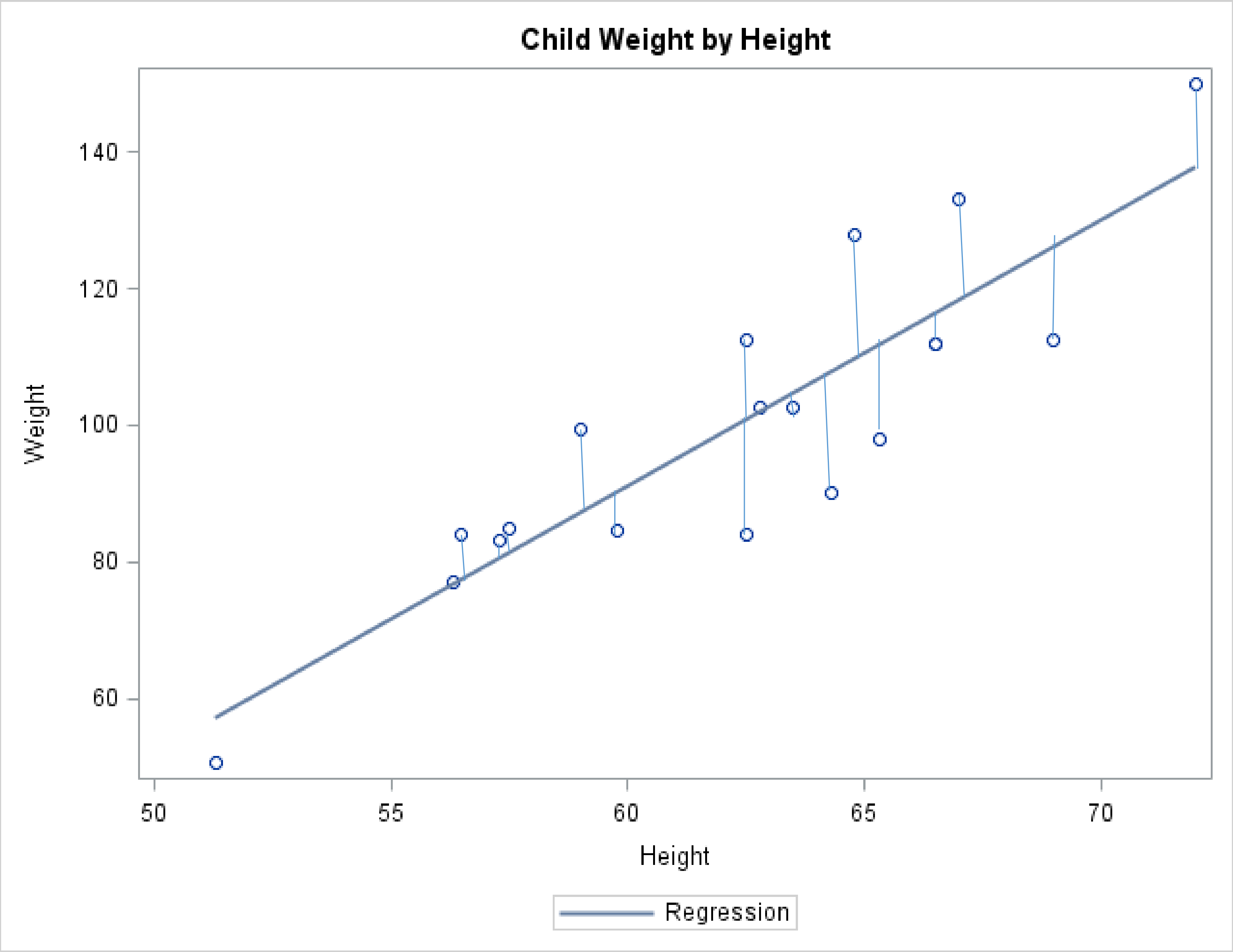
A Simple Linear Regression Example



Data from Lewis and Taylor (1967) via
http://support.sas.com/documentation/cdl/en/statug/68162/HTML/default/viewer.htm#statug_reg_examples03.htm

Goal: Find straight line that minimizes sum of squared distances from actual weight to fitted line

“Least squares fit”



A Simple Linear Regression Example—SAS Code

```
proc reg data = Children;
  model Weight = Height;
run;
```

Children is a SAS dataset including variables Weight and Height

Simple Linear Regression Example—SAS Output

S.E. of slope and intercept

Parameter estimates divided by S.E.

P-Values

Intercept: Estimated weight for child of height 0
(Not always interpretable…)

Parameter Estimates					
Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Intercept	1	-143.02692	32.27459	-4.43	0.0004
Height	1	3.89903	0.51609	7.55	<.0001

Slope: How much weight increases for a 1 inch increase in height

Weight increases significantly with height

Weight = -143.0 + 3.9*Height

Simple Linear Regression Example—SAS Output

Sum of squared differences between model fit and mean of Y

Sum of squares/df

Mean Square(Model)/MSE

Analysis of Variance					
Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	1	7193.24912	7193.24912	57.08	<.0001
Error	17	2142.48772	126.02869		
Corrected Total	18	9335.73684			

Sum of squared differences between model fit and observed values of Y

Regression on X provides a significantly better fit to Y than the null (intercept-only) model

Sum of squared differences between mean of Y and observed values of Y

Simple Linear Regression Example—SAS Output

Percent of variance of Y explained by regression

Root MSE	11.22625	R-Square	0.7705
Dependent Mean	100.02632	Adj R-Sq	0.7570
Coeff Var	11.22330		

Mean of Y

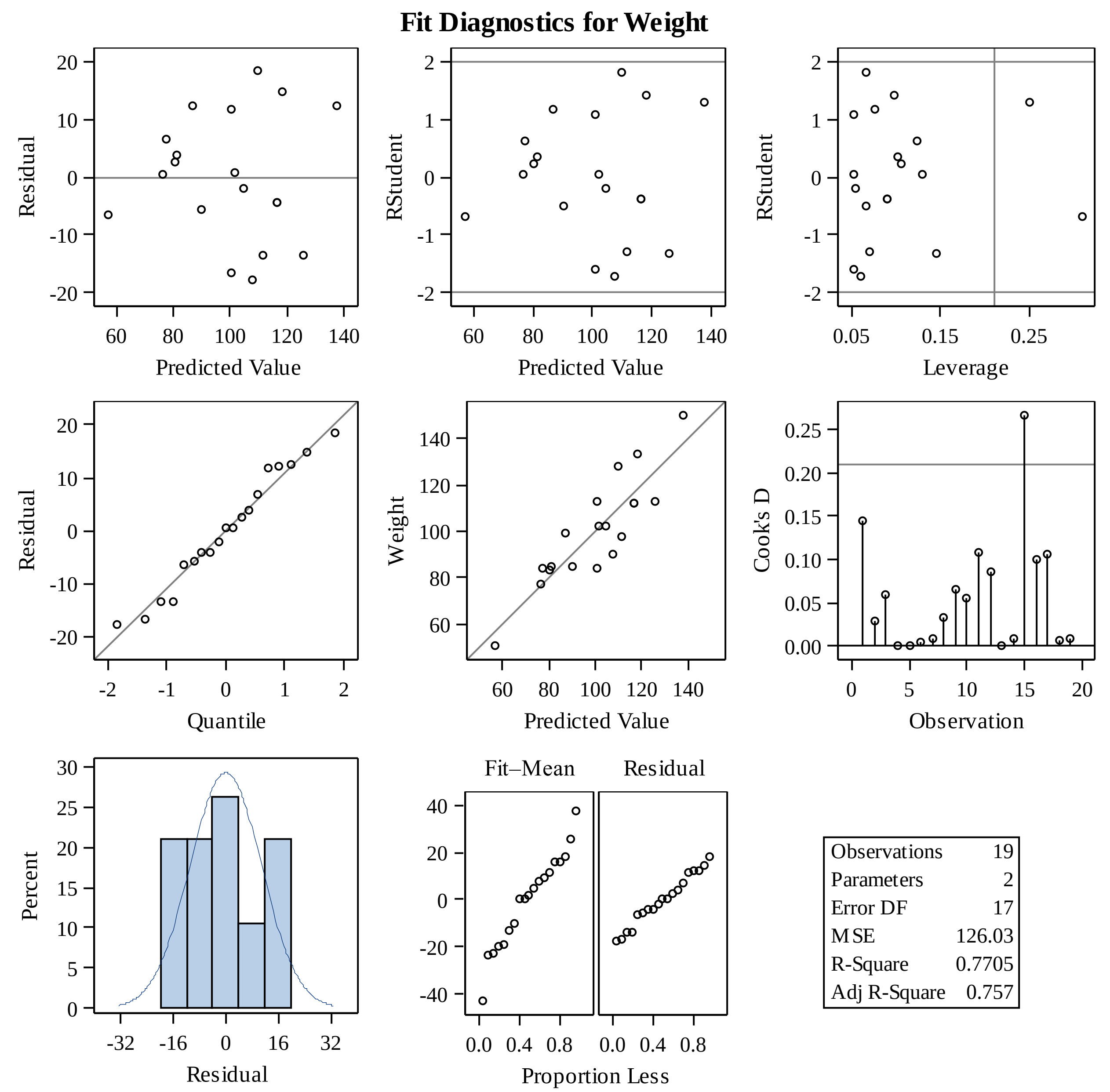
Root MSE/mean of Y

Version of R-square adjusted for number of predictors in model

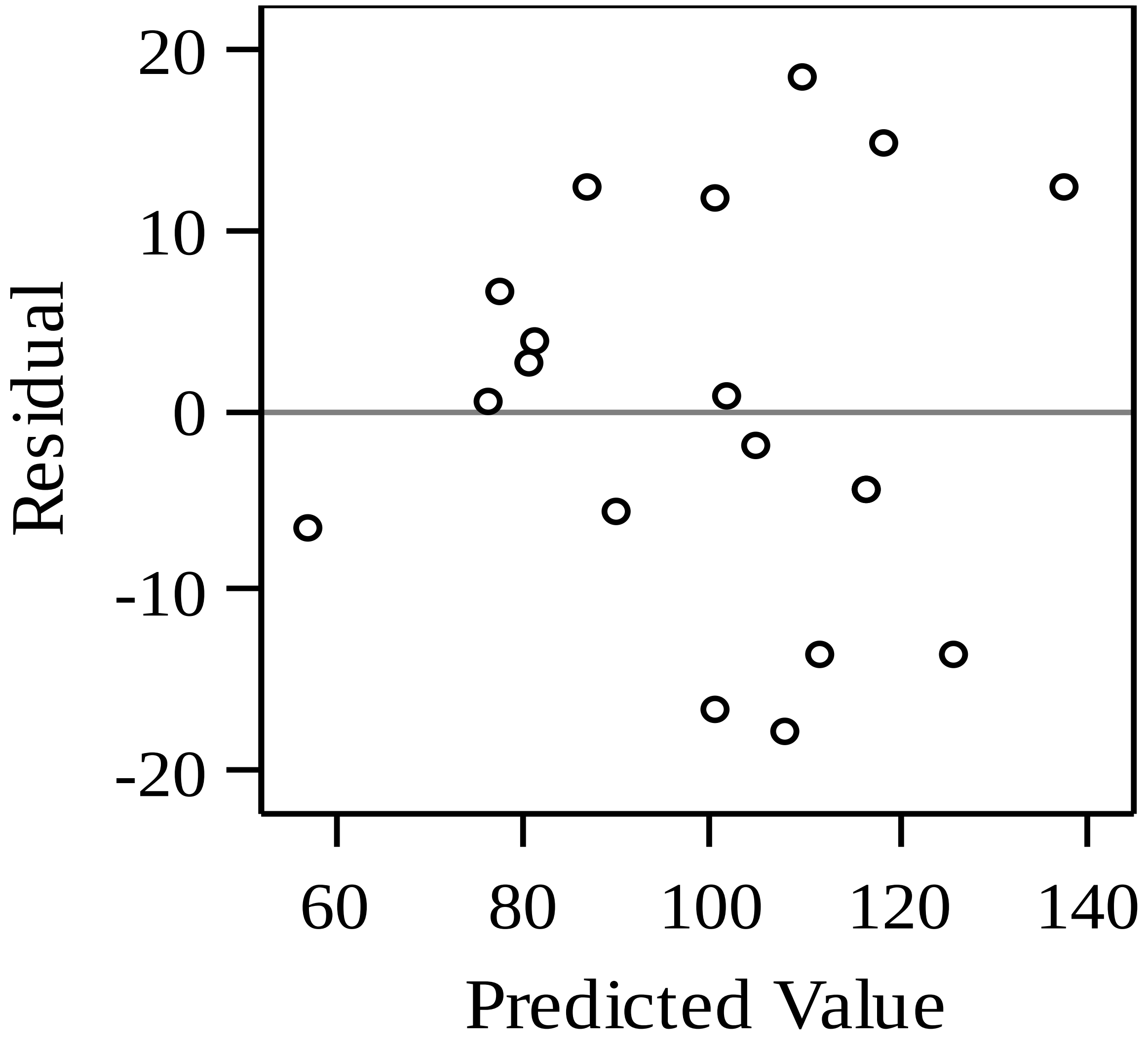
Thoughts on R-Squared

- For our model, R-square is **`0.7705`**
- 77% of the variability in weight is explained by height
- Not a measure of goodness of fit of the model:
  - If variance is high, will be low even with the “right” model
  - Can be high with “wrong” model (e.g. Y isn’t linear in X)
  - See http://data.library.virginia.edu/is-r-squared-useless/
- Always gets higher when you add more predictors
  - Adjusted R-square intended to correct for this
- Take with a grain of salt

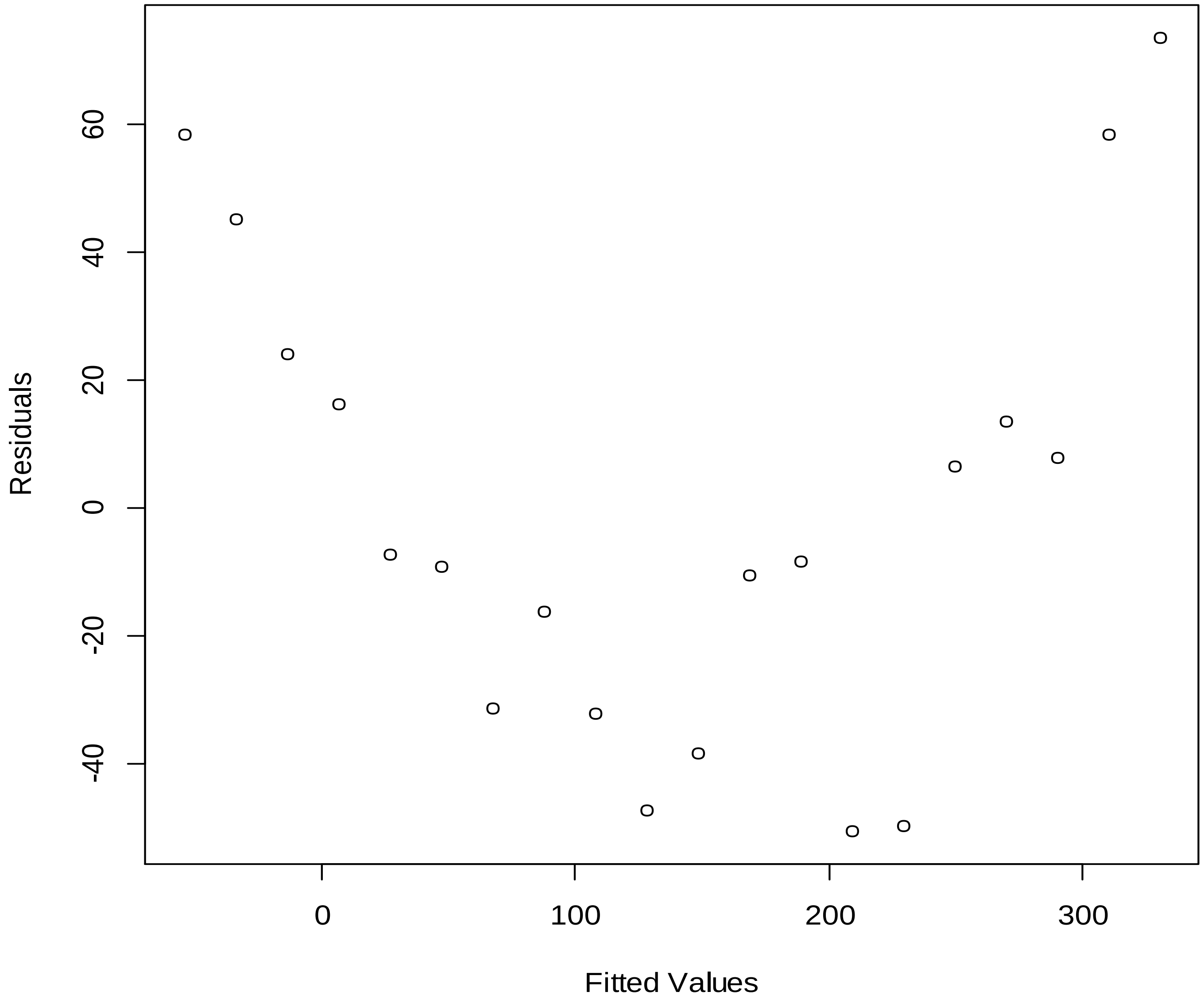
Simple Linear Regression Example—SAS Output



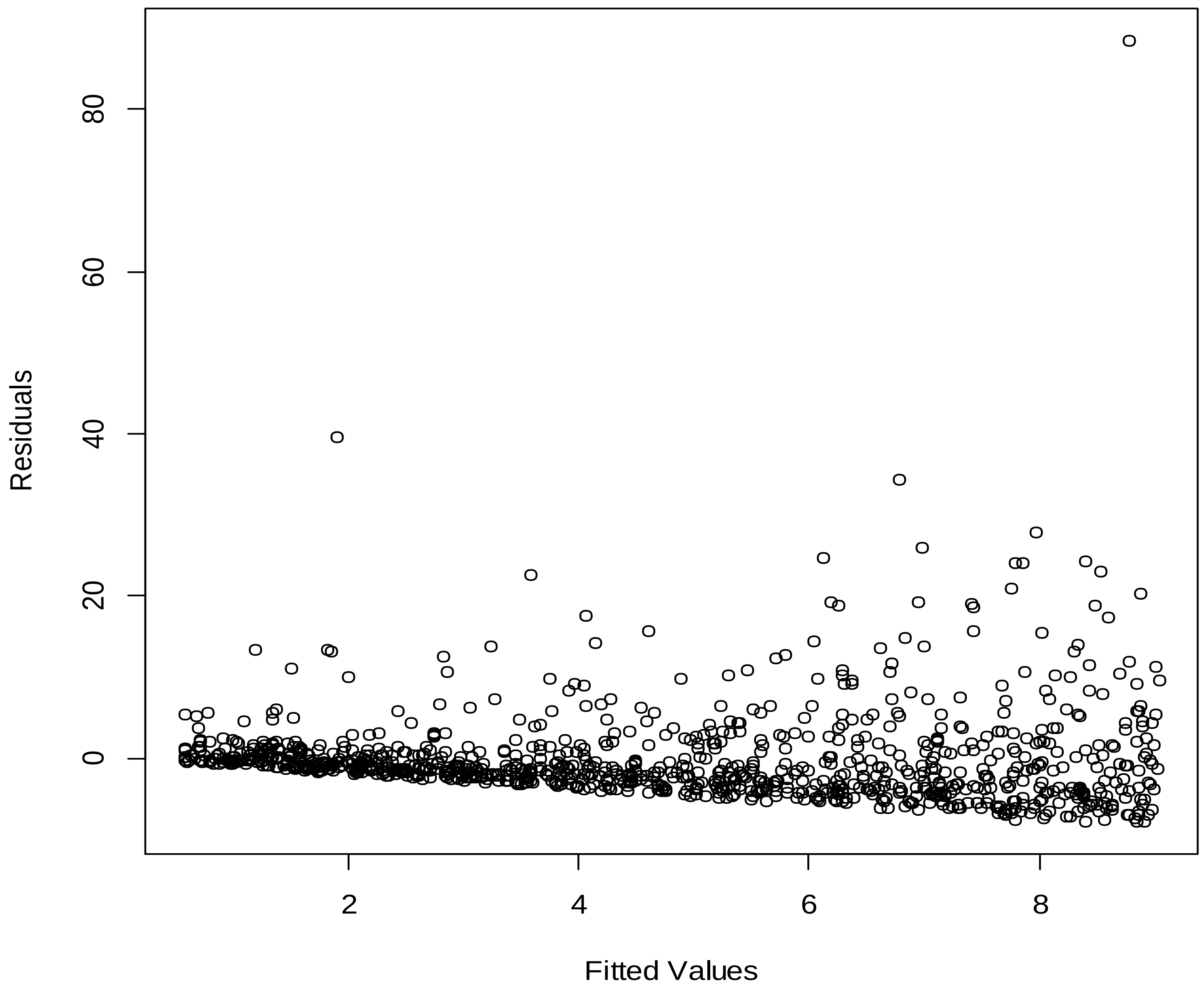
Observations	19
Parameters	2
Error DF	17
MSE	126.03
R-Square	0.7705
Adj R-Square	0.757



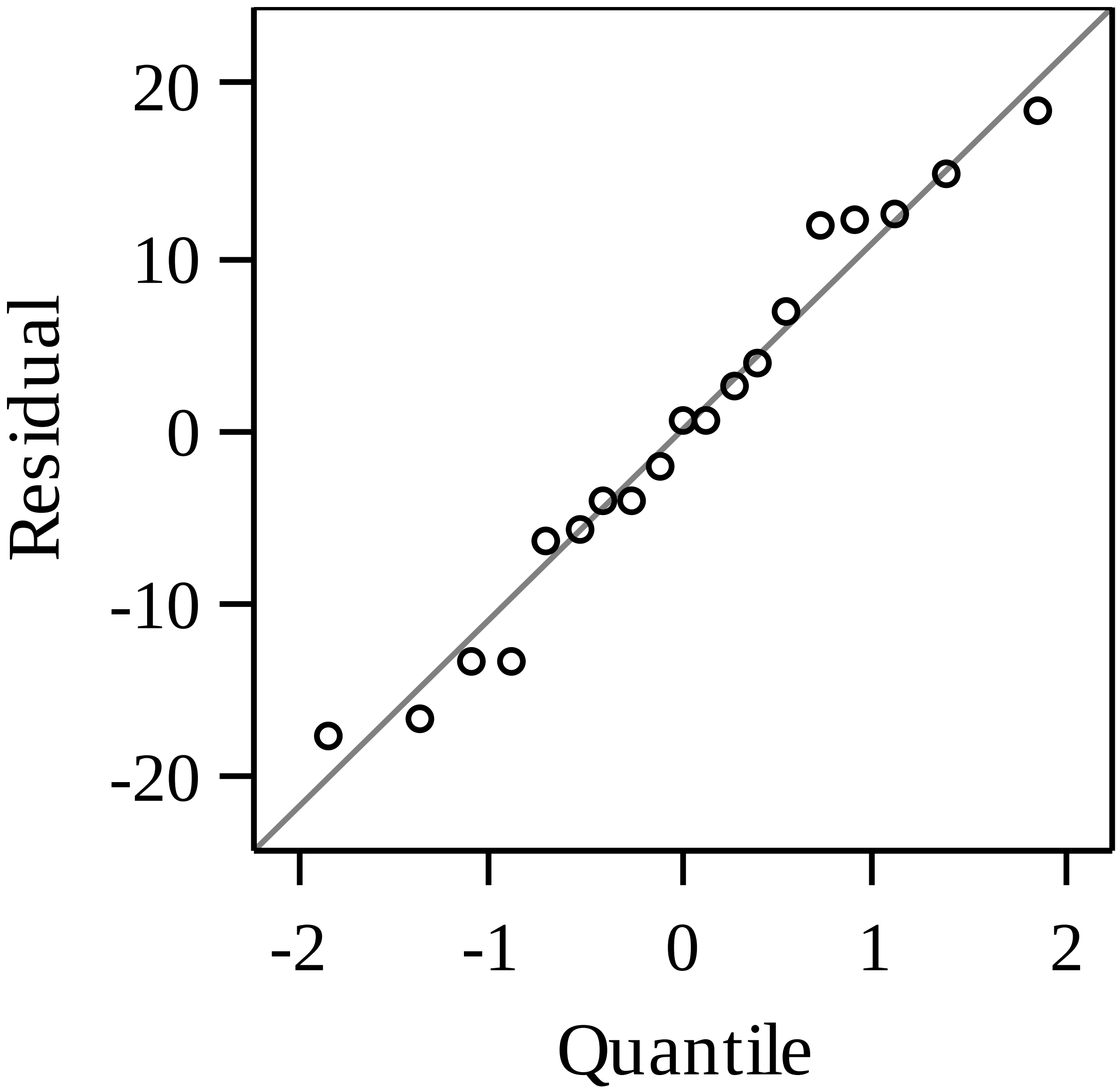
- Residuals should form even band around 0
- Size of residuals shouldn’t change with predicted value
- Sign of residuals shouldn’t change with predicted value



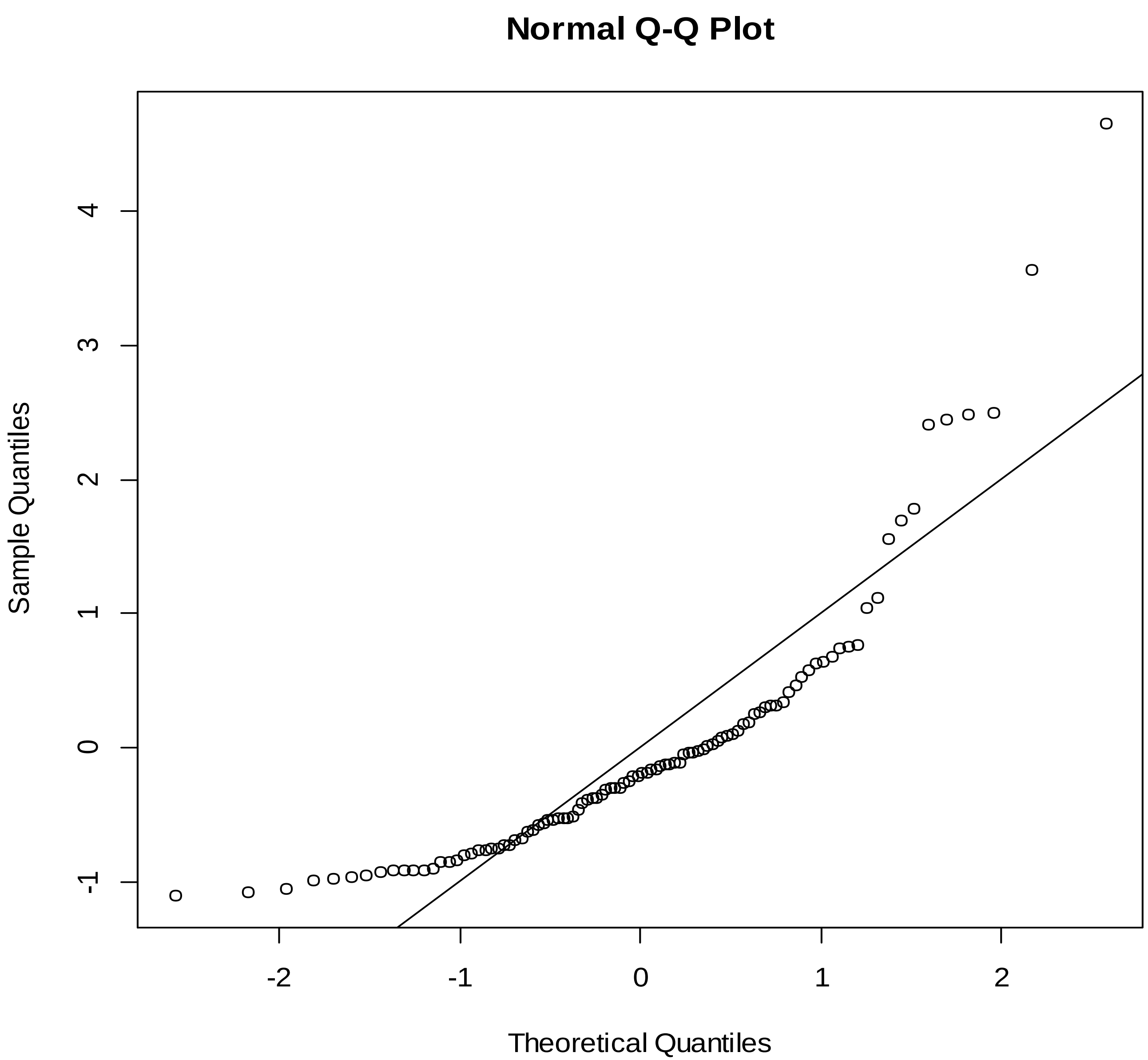
Suggests Y and X have a nonlinear relationship



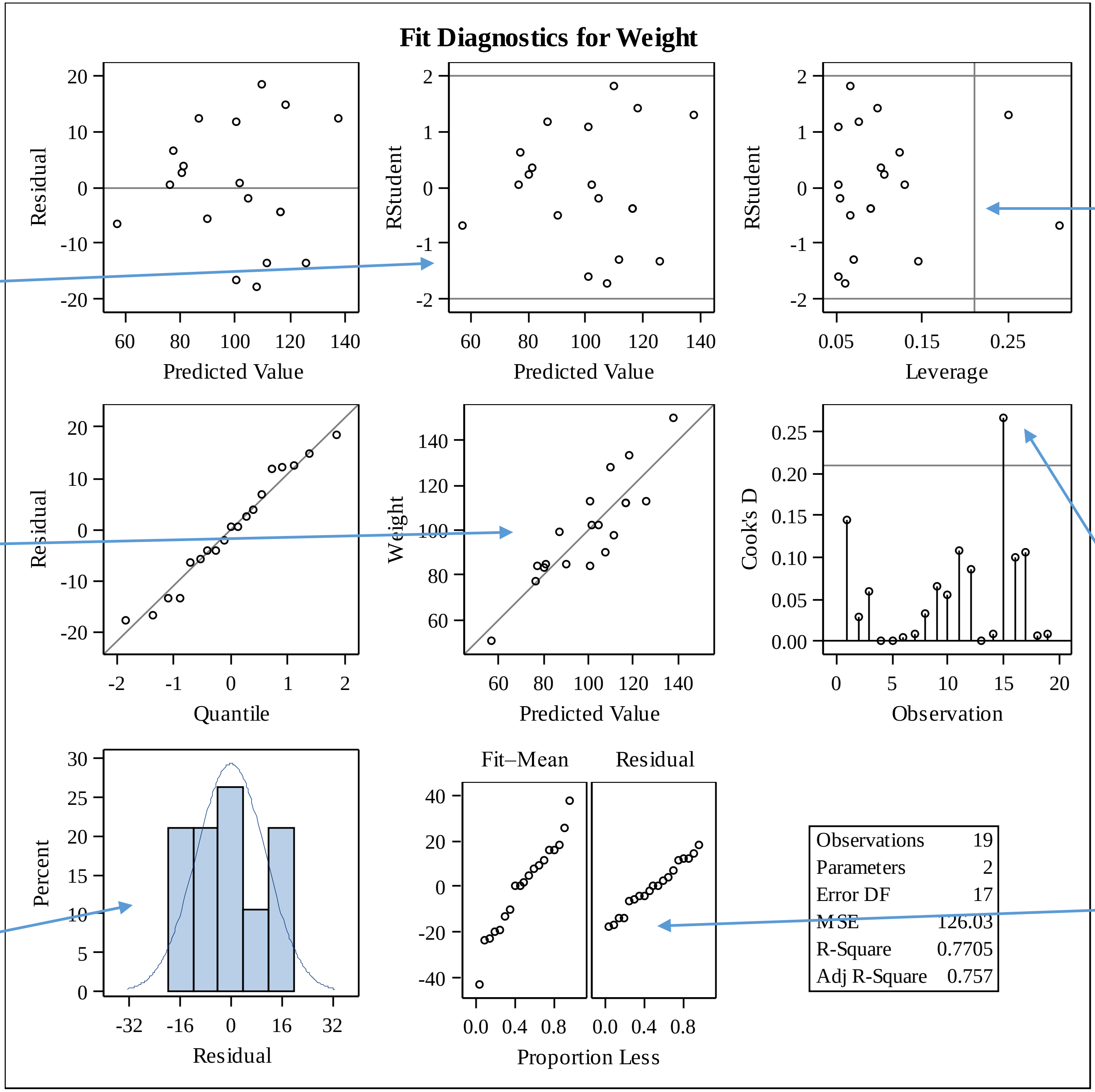
Suggests data transformation



- Plot of model residuals versus quantiles of a normal distribution
- Deviations from diagonal line suggest departures from normality



Suggests data transformation may be needed



Studentized (scaled) residuals by predicted values (cutoff for outlier depends on n, use 3.5 for n = 19 with 1 predictor)

Studentized residuals by leverage, leverage > 2(p + 1)/n (= 0.21) suggests influential observation

Y by predicted values (should form even band around line)

Cook's distance > 4/n (= 0.21) may suggest influence (cutoff of 1 also used)

Histogram of residuals (look for skewness, other departures from normality)

Residual-fit plot, see Cleveland, *Visualizing Data* (1993)

Thoughts on Outliers

- An outlier is NOT a point that fails to support the study hypothesis
- Removing data can introduce biases
- Check for outlying values in X and Y before fitting model, not after
- Is there another model that fits better? Do you need a nonlinear model or data transformation?
- Was there an error in data collection?
- Robust regression is an alternative

Multiple Linear Regression

A Multiple Linear Regression Example—SAS Code

```
proc reg data = Children;
  model Weight = Height Age;
run;
```

A Multiple Linear Regression Example—SAS Output

Parameter Estimates					
Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Intercept	1	-141.22376	33.38309	-4.23	0.0006
Height	1	3.59703	0.90546	3.97	0.0011
Age	1	1.27839	3.11010	0.41	0.6865

Adjusting for age, weight still increases significantly with height (P = 0.0011).
Adjusting for height, weight is not significantly associated with age (P = 0.6865)

Categorical Variables

- Let’s try adding in gender, coded as “M” and “F”:

```
proc reg data = Children;
    model Weight = Height Gender;
run;
```

```
ERROR: Variable Gender in list does not match
type prescribed for this list.
```

Categorical Variables

- For proc reg, categorical variables have to be recoded as 0/1 variables:

```
data children;
     set children;
     if Gender = 'F' then numgen = 1;
     else if Gender = 'M' then numgen = 0;
     else call missing(numgen);
run;
```

Categorical Variables

- Let’s try fitting our model with height and gender again, with gender coded as 0/1:

```
proc reg data = Children;
    model Weight = Height numgen;
run;
```

Categorical Variables

Parameter Estimates					
Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Intercept	1	-126.16869	34.63520	-3.64	0.0022
Height	1	3.67890	0.53917	6.82	<.0001
numgen	1	-6.62084	5.38870	-1.23	0.2370

Adjusting for gender, weight still increases significantly with height

Adjusting for height, mean weight does not differ significantly between genders

Categorical Variables

- Can use proc glm to avoid recoding categorical variables:
- Recommend this approach if a categorical variable has more than 2 levels

```
proc glm data = children;
    class Gender;
    model Weight = Height Gender;
run;
```

Proc glm output

Source	DF	**Type I SS**	Mean Square	F Value	Pr > F
Height	1	7193.249119	7193.249119	58.79	<.0001
Gender	1	184.714500	184.714500	1.51	0.2370

Source	DF	**Type III SS**	Mean Square	F Value	Pr > F
Height	1	5696.840666	5696.840666	46.56	<.0001
Gender	1	184.714500	184.714500	1.51	0.2370

- Type I SS are sequential
- Type III SS are nonsequential

Proc glm

- By default, proc glm only gives ANOVA tables
- Need to add estimate statement to get parameter estimates:

```
proc glm data = children;
    class Gender;
    model Weight = Height Gender;
    estimate 'Height' height 1;
    estimate 'Gender' Gender 1 -1;
run;
```

Proc glm

Parameter	Estimate	Standard Error	t Value	Pr > |t|
Height	3.67890306	0.53916601	6.82	<.0001
Gender	-6.62084305	5.38869991	-1.23	0.2370

Same estimates as with proc reg

Model Selection

- ‘Rule of thumb’ suggests model should include no more than 1 covariate for every 10—15 observations
- What if you have more?
  - Pre-specify a smaller model based on literature, subject-matter knowledge, etc.
  - Select a smaller model in a data-driven fashion

Stepwise Methods

- Forward selection: Start with best single-variable model, add variables until no variable meets criteria to enter model
- Backward elimination: Start with full model, remove variables until no variable meets criteria to be removed
- Forward and backward selection: Variables can be added and removed at each step

Stepwise Methods

- Different criteria to enter and leave model can be used:
  - P-value from ANOVA F-test
  - Mallows C(p)
    - Estimate of mean square prediction error
  - Adjusted R^2
  - AIC (not implemented in proc reg)

Model Selection Example

Data Set Name	WORK.NSQIP_BASECHARS	Observations	1413
Member Type	DATA	Variables	7
Engine	V9	Indexes	0
Created	04/08/2017 14:24:17	Observation Length	56
Last Modified	04/08/2017 14:24:17	Deleted Observations	0
Protection		Compressed	NO
Data Set Type		Sorted	NO
Label			
Data Representation	WINDOWS_64		
Encoding	wlatin1 Western (Windows)		

Alphabetic List of Variables and Attributes			
#	Variable	Type	Len
1	age2	Num	8
6	album	Num	8
7	bmi	Num	8
5	creat	Num	8
4	sex	Num	8
3	smoke	Num	8
2	steroid	Num	8

Model Selection Example

```
proc reg data = nsqip_basechars;
    model logcreat = bmi logalbum steroid
smoke age2 sex / selection = forward;
run;
```

Model Selection Example

Summary of Forward Selection							
Step	Variable Entered	Number Vars In	Partial R-Square	Model R-Square	C(p)	F Value	Pr > F
1	sex	1	0.0842	0.0842	57.7965	129.68	<.0001
2	age2	2	0.0305	0.1147	11.0253	48.50	<.0001
3	logalbum	3	0.0059	0.1206	3.6103	9.42	0.0022
4	smoke	4	0.0013	0.1219	3.4948	2.12	0.1458

Model Selection Example

```
proc reg data = nsqip_basechars;
    model logcreat = bmi logalbum steroid
smoke age2 sex / selection = backward;
run;
```

Model Selection Example

Summary of Backward Elimination							
Step	Variable Removed	Number Vars In	Partial R-Square	Model R-Square	C(p)	F Value	Pr > F
1	steroid	5	0.0001	0.1221	5.2208	0.22	0.6385
2	bmi	4	0.0002	0.1219	3.4948	0.27	0.6006
3	smoke	3	0.0013	0.1206	3.6103	2.12	0.1458

Variable	Parameter Estimate	Standard Error	Type II SS	F Value	Pr > F
Intercept	-0.41900	0.07742	2.17950	29.29	<.0001
logalbum	0.14193	0.04625	0.70071	9.42	0.0022
age2	0.00345	0.00047777	3.88588	52.23	<.0001
sex	-0.17584	0.01473	10.60565	142.54	<.0001

Caveats about stepwise model selection

- Test statistics of final model don’t have the correct distributions
  - P-values will be incorrect
- Regression coefficients will be biased
- R-squared values will be too high
- Doesn’t handle multicollinearity well

See http://www.stata.com/support/faqs/statistics/stepwise-regression-problems/ among many others

Multicollinearity

- Highly correlated covariates cause problems
  - Inflated standard errors
  - Sometimes can’t fit model at all
- Two highly correlated variables might be significant on their own and very non-significant when included in a model together

Diagnosing Multicollinearity

```
proc reg data = nsqip_basechars;
    model logcreat = logalbum smoke
age2 sex / vif;
run;
```

Diagnosing Multicollinearity

<table>
<tr><th colspan="7">Parameter Estimates</th></tr>
<tr><th>Variable</th><th>DF</th><th>Parameter Estimate</th><th>Standard Error</th><th>t Value</th><th>Pr > |t|</th><th>Variance Inflation</th></tr>
<tr><td>Intercept</td><td>1</td><td>-0.39537</td><td>0.07907</td><td>-5.00</td><td><.0001</td><td>0</td></tr>
<tr><td>logalbum</td><td>1</td><td>0.13626</td><td>0.04639</td><td>2.94</td><td>0.0034</td><td>1.01490</td></tr>
<tr><td>smoke</td><td>1</td><td>-0.02821</td><td>0.01939</td><td>-1.46</td><td>0.1458</td><td>1.06614</td></tr>
<tr><td>age2</td><td>1</td><td>0.00328</td><td>0.00049220</td><td>6.66</td><td><.0001</td><td>1.07280</td></tr>
<tr><td>sex</td><td>1</td><td>-0.17541</td><td>0.01473</td><td>-11.91</td><td><.0001</td><td>1.00267</td></tr>
</table>

Variance Inflation Factor (VIF): Rule of thumb suggests VIF > 10 means severe multicollinearity

Other Issues

Data Transformations

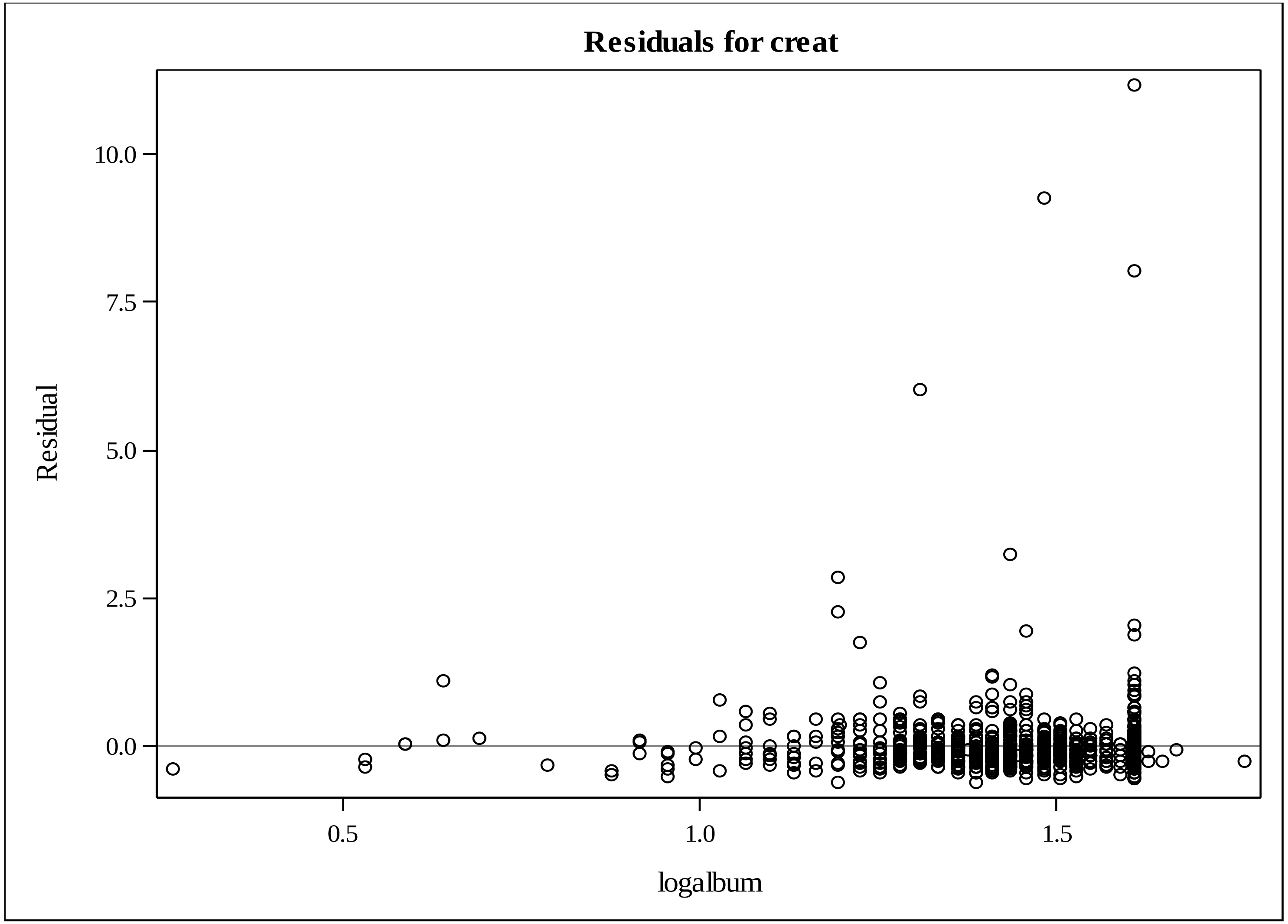
- Skewed residuals
- Residuals with non-constant variance
- Large ‘outliers’ at one end of the data scale

- Data transformation may be the answer to these issues

Data Transformations

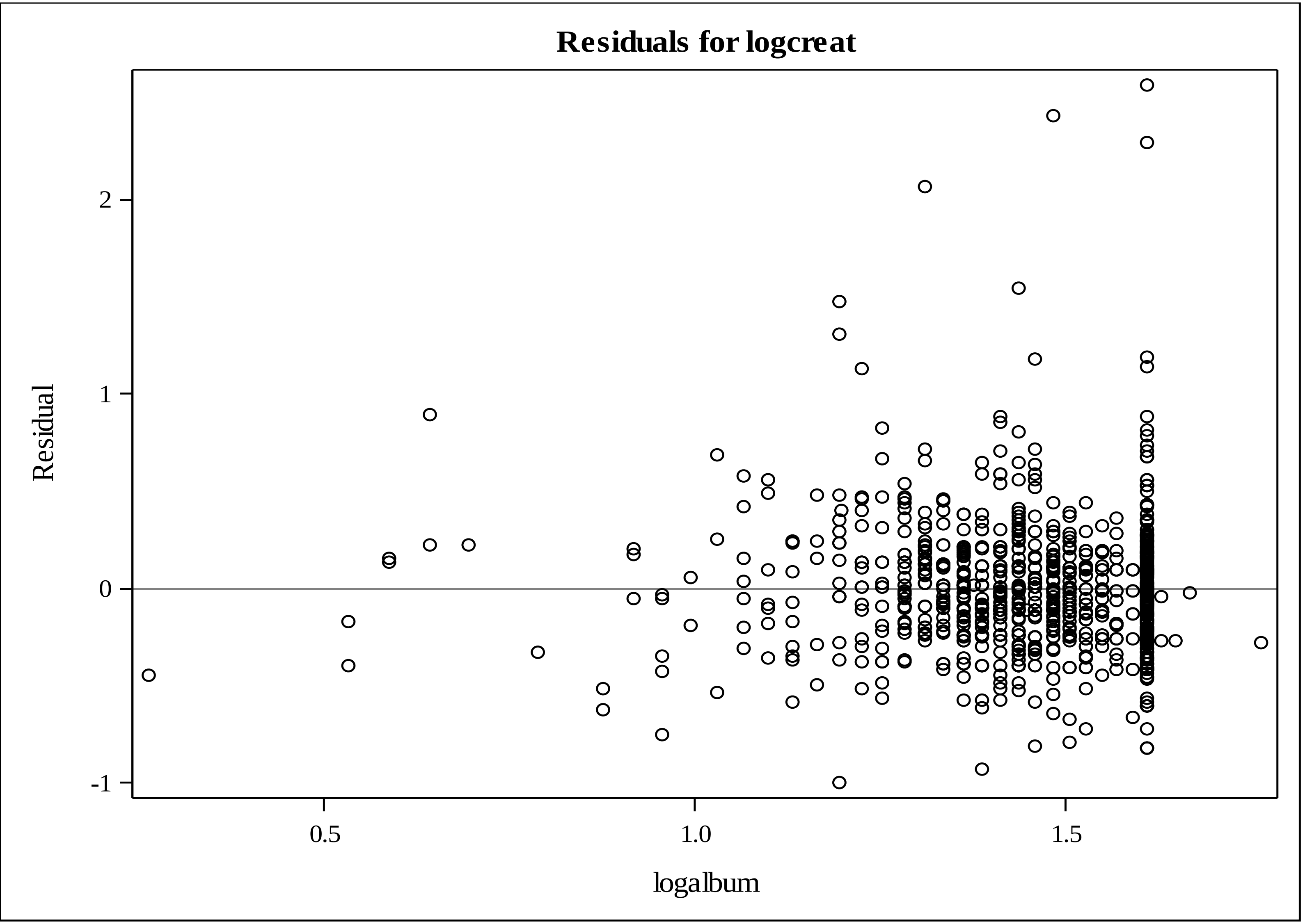
- Log transformation: Useful for lab data, other biological assay data
- Reciprocal transformation: Reduces skewness
- Logit transformation: Use with percentages bounded away from 0 and 1
- Box-Cox family of power transformations

Data Transformation Example



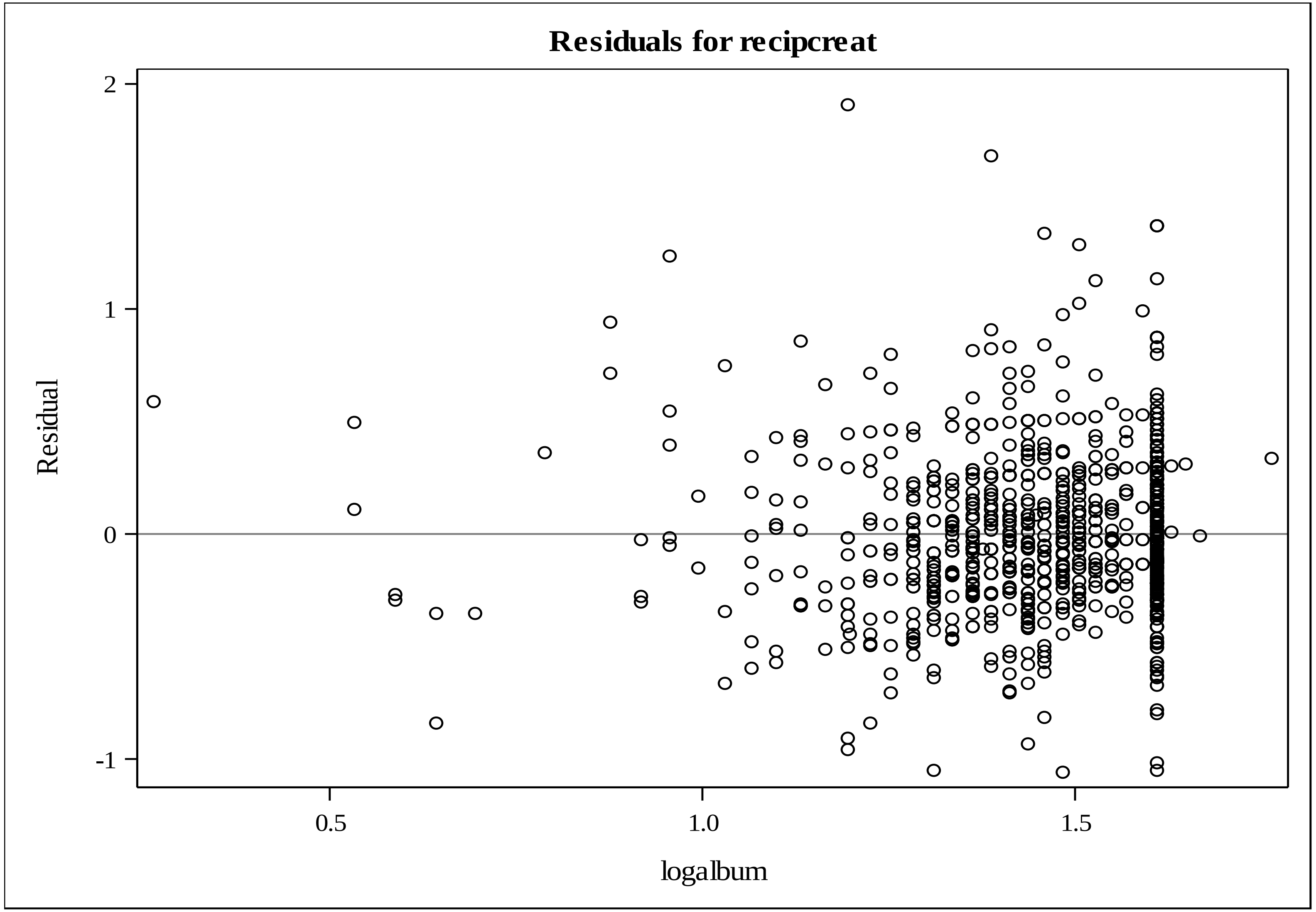

```
proc reg data = nsqip_basechars;
      model creat = logalbum;
run;
```

Data Transformation Example




```
proc reg data = nsqip_basechars;
        model logcreat = logalbum;
run;
```

Data Transformation Example




```
proc reg data =
nsqip_basechars;
        model recipcreat =
logalbum;
run;
```

Regression vs. Correlation

- Regression and correlation analysis are closely related
- Regression designates one variable as the outcome, correlation does not
- Regression slope = Pearson correlation X SD(Y)/SD(X)
- P-values from simple linear regression and from correlation test will be identical

Thank you!

Help is Available

- **CTSC Biostatistics Office Hours**
  - Every Tuesday from 12 – 1:30 in Sacramento
  - Sign-up through the CTSC Biostatistics Website
- **EHS Biostatistics Office Hours**
  - Every Monday from 2-4 in Davis
- **Request Biostatistics Consultations**
  - CTSC - www.ucdmc.ucdavis.edu/ctsc/
  - MIND IDDRC - www.ucdmc.ucdavis.edu/mindinstitute/centers/iddrc/cores/bbrd.html
  - Cancer Center and EHS Center